

MULTI-HAMILTONIAN STRUCTURE FOR THE FINITE DEFOCUSING ABLOWITZ-LADIK EQUATION

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ABSTRACT. We study the Poisson structure associated to the defocusing Ablowitz-Ladik equation from a functional-analytical point of view, by reexpressing the Poisson bracket in terms of the associated Carathéodory function. Using this expression, we are able to introduce a family of compatible Poisson brackets which form a multi-Hamiltonian structure for the Ablowitz-Ladik equation. Furthermore, we show using some of these new Poisson brackets that the Geronimus relations between orthogonal polynomials on the unit circle and those on the interval define an algebraic and symplectic mapping between the Ablowitz-Ladik and Toda hierarchies.

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1. INTRODUCTION

It has been well-known since the work of Flaschka [Fla1], [Fla2] that the celebrated Toda lattice can be represented as an isospectral evolution equation for Jacobi matrices, by which we mean symmetric, tridiagonal matrices J , with positive off-diagonals. This not only allowed Flaschka and Hénon to prove the complete integrability of the Toda equation, but also opened the door for the study of related Lie algebraic and geometric structures for generalized Toda flows with applications ranging from numerical analysis to quantum cohomology. We will not attempt to properly cite vast literature on the Toda lattice, instead referring the reader to the bibliography in [OPRS] or in the recent survey [BloGek]. We will mention, however, two seminal works that introduced two aspects of the theory crucial to our exposition: Moser's paper [Mos] that explained the role played by spectral data in both explicitly solving and establishing complete integrability of the Toda lattice and Kostant's comprehensive treatment of the generalized Toda lattice as a restriction of the larger Hamiltonian system to a minimal irreducible coadjoint orbit of the Borel subgroup (the manifold of Jacobi matrices with fixed trace in the case of $gl(n)$) equipped with the Lie-Poisson structure. Combination of these two approaches later allowed Deift *et al.* to establish complete integrability of full Toda flows [DLNT1]. More recently, it was shown in [FayGek2], that the spectral data can be used to define in a natural way a multi-Hamiltonian structure for a family of Toda-like systems associated with all minimal irreducible coadjoint orbits in $gl(n)$.

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In this paper we concentrate on the study of another integrable system, the defocusing Ablowitz-Ladik (AL) hierarchy, through its connection to a unitary analogue of Jacobi matrices, the so-called CMV matrices. The defocusing AL equation was defined in 1975–76 by Ablowitz and Ladik [AblLad1, AblLad2] as a space-discretization of the cubic nonlinear Schrödinger equation. It reads:

$$(1.1) \quad -i\dot{\beta}_k = \rho_k^2(\beta_{k+1} + \beta_{k-1}) - 2\beta_k,$$

where $\beta = \{\beta_k\} \subset \mathbb{D}$ is a sequence of complex numbers inside the unit disk and $\rho_k^2 = 1 - |\beta_k|^2$. The analogy with the continuous NLS becomes transparent if we rewrite (1.1) as¹

$$-i\dot{\beta}_k = \beta_{k+1} - 2\beta_k + \beta_{k-1} - |\beta_k|^2(\beta_{k+1} + \beta_{k-1}).$$

A simple change of variables, $\alpha_k = e^{2it}\beta_k$ for all k , transforms (1.1) into

$$(1.2) \quad -i\dot{\alpha}_k = \rho_k^2(\alpha_{k+1} + \alpha_{k-1}),$$

where $\rho_k = \sqrt{1 - |\alpha_k|^2}$. This is the equation we will refer to as Ablowitz-Ladik (or AL). In this paper we focus on the finite case, by which we mean the case in which $\alpha_{-1} = -1$ and, for some fixed $n \geq 1$, $\alpha_{n-1} \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. These are Dirichlet boundary conditions, and one can easily see from (1.2) that the evolution for $\alpha_0, \dots, \alpha_{n-2}$ decouples from the evolution for the other α 's. Consider the following Poisson bracket on the space of $(\alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1}) \in \mathbb{D}^{n-1} \times S^1$:

$$(1.3) \quad \{f, g\} = i \sum_{k=0}^{N-2} \rho_k^2 \left[\frac{\partial f}{\partial \bar{\alpha}_k} \frac{\partial g}{\partial \alpha_k} - \frac{\partial f}{\partial \alpha_k} \frac{\partial g}{\partial \bar{\alpha}_k} \right],$$

where $\rho_k = \sqrt{1 - |\alpha_k|^2}$ and, for a complex variable $\alpha = u + iv$, $u, v \in \mathbb{R}$, the partial derivatives are defined as usual by

$$\frac{\partial}{\partial \alpha} = \frac{1}{2} \left[\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right], \quad \frac{\partial}{\partial \bar{\alpha}} = \frac{1}{2} \left[\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right].$$

In this Poisson structure, the AL equation (1.2) becomes completely integrable. Moreover, it can be re-written in the Lax form, with the Hamiltonian $\text{Re Tr } \mathcal{C}$, where the Lax operator \mathcal{C} is the CMV matrix associated with the coefficients $\alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1}$ (for the background, see Subsection 2.1). In fact, one can define a whole hierarchy of evolution equations, that we will call the AL hierarchy, by considering the Hamiltonians given by the real and imaginary parts of $K_k = \frac{1}{k} \text{Tr}(\mathcal{C}^k)$ for $k \geq 1$. In terms of Lax pairs, the hierarchy is given by the following evolution equations (see [Nen1]):

$$(1.4) \quad \{\mathcal{C}, 2 \text{Re}(K_k)\} = [\mathcal{C}, i(\mathcal{C}^k)_+ + i((\mathcal{C}^k)_+)^*]$$

and

$$(1.5) \quad \{\mathcal{C}, 2 \text{Im}(K_k)\} = [\mathcal{C}, (\mathcal{C}^k)_+ - ((\mathcal{C}^k)_+)^*]$$

for all $k \geq 1$, where for a matrix X , we have

$$(X_+)^{jk} = \begin{cases} X_{jk}, & \text{if } j < k; \\ \frac{1}{2}X_{jj}, & \text{if } j = k; \\ 0, & \text{if } j > k. \end{cases}$$

One of the central ingredients in the study of the AL hierarchy is, similarly to the Toda case, rewriting the Poisson bracket (1.3) as the restriction to the manifold of CMV matrices of the Gelfand-Dikij bracket on the associative algebra $M_n(\mathbb{C})$ of $n \times n$ matrices (for a short background, see Subsection 2.2). This was done independently by L. C. Li [Li], and R. Killip and I. Nenciu [KilNen2], and allowed Killip and Nenciu to solve the system and find the long-time asymptotics of the α s, and of certain associated spectral quantities. Inverse spectral problem for semi-infinite CMV matrices was recently utilized in this context in [Gol]. Note also that an alternative Lax representation was used to linearize finite and semi-infinite AL flows in [Gek], while the approach based on continued fractions was suggested in [Com].

¹Here, and throughout the paper, \dot{f} will denote the time derivative of the function f .

If $d\mu$ is the measure on the unit circle associated to the Verblunsky coefficients $(\alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1}) \in \mathbb{D}^{n-1} \times S^1$, then it is known (see Subsection 2.1) that $d\mu = \sum_{j=1}^n \mu_j \delta_{z_j}$, with $\mu_j \in (0, 1)$, $\mu_1 + \dots + \mu_n = 1$, and $z_j = e^{i\theta_j} \in S^1$ for all $1 \leq j \leq n$. The function

$$\prod_{j=1}^n z_j = \det(\mathcal{C}) = (-1)^{n-1} \bar{\alpha}_{n-1}$$

is a Casimir, and the manifold of CMV matrices with fixed determinant forms a symplectic leaf on which, for $1 \leq j, k \leq n-1$ ², we have

$$(1.6) \quad \{\theta_j, \theta_k\} = 0, \quad \{\theta_j, \tfrac{1}{2} \log[\mu_k/\mu_n]\} = \delta_{jk},$$

and

$$(1.7) \quad \{\log[\mu_j/\mu_n], \log[\mu_k/\mu_n]\} = 2 \cot\left(\frac{\theta_j - \theta_k}{2}\right) + 2 \cot\left(\frac{\theta_k - \theta_n}{2}\right) + 2 \cot\left(\frac{\theta_n - \theta_j}{2}\right).$$

These results are obtained by regarding the CMV matrices, or, equivalently, the associated spectral measures, as the central objects. In this paper, we adopt a slightly different point of view, and focus on the associated *Carathéodory function*,

$$F(z) = \left(\frac{\mathcal{C} + z}{\bar{\mathcal{C}} - z} \right)_{11}.$$

This is the analogue in the unitary case of the better known *Weyl function* (or *m-function*) associated to a Jacobi matrix,

$$m(\lambda) = \left(\frac{1}{J - \lambda} \right)_{11}.$$

In [FayGek2], Faybusovich and Gekhtman adopted this point of view for the Toda lattice and computed Poisson brackets induced by the Lie-Poisson structure for $m(\lambda)$ and $m(\xi)$ at any two distinct points λ and ξ . The resulting Poisson structure on Weyl functions was then shown to be a part of a family of compatible Poisson brackets which constitutes a multi-Hamiltonian structure for the Toda lattice. In this paper, we follow the same road in the Ablowitz-Ladik case.

The paper is organized as follows. In Section 2 we give some background information on the theory of orthogonal polynomials on the real line and unit circle, and on classical *R*-matrices. Section 3 contains the first important results, Theorem 3 and Corollary 3.3, which gives the formula for the Poisson bracket of the Carathéodory function at two distinct points in the complex plane. In particular, this represents a more direct proof of some of the results in [KilNen2]. Here we should also mention that in a recent paper [CanSim], M. Cantero and B. Simon study Poisson brackets for orthogonal polynomials both on the real line and on the unit circle induced by *standard* Poisson structures for the Toda and Ablowitz-Ladik hierarchies, respectively. An essential part of their analysis are the formulae for the Poisson brackets of Weyl and Carathéodory functions, for which they give new proofs by induction, using purely orthogonal polynomial methods. Another related recent paper is [Tsi], which also introduces a family of Poisson brackets compatible with the Sklyanin bracket associated with the standard 2×2 *rational* solution of the Classical Yang-Baxter equation. These Poisson brackets are defined on monodromy matrices associated with 2×2 spectral parameter depending Lax representation for a family of integrable systems that includes both open and periodic Toda lattices.

The formula (3.12) for the Poisson bracket mentioned above allows us to extend the Poisson structure to the space of finite, but unnormalized measures on the circle, and obtain the canonical coordinates for both the extended and the usual Poisson structures; this is achieved in Section 4. Finally, we define the family of compatible Poisson structures in Section 5, and show its connection to the defocusing Ablowitz-Ladik equation. In the last section, Section 6, we go back to the Toda lattice and show that, loosely speaking, ‘half’ of the Ablowitz-Ladik hierarchy (also known as the Schur flows) is mapped symplectically onto the Toda lattice hierarchy via the well-known Geronimus relations.

²Since the z_j s are always distinct, any choice of labeling for z_1, \dots, z_n is locally well-defined and leads to these formulae.

2. BACKGROUND

2.1. Orthogonal polynomials. As CMV matrices arose in the study of orthogonal polynomials, it is natural that we begin there. We will first describe the relation of orthogonal polynomials to Jacobi matrices and then explain the connection to CMV matrices.

Given a probability measure $d\nu$ supported on a finite subset of \mathbb{R} , say of cardinality n , we can apply the Gram–Schmidt procedure to $\{1, x, x^2, \dots, x^{n-1}\}$ and so obtain an orthonormal basis for $L^2(d\nu)$ consisting of polynomials, $\{p_j(x) : j = 0, \dots, n-1\}$, with positive leading coefficient. In this basis, the linear transformation $f(x) \mapsto xf(x)$ is represented by a Jacobi matrix,

$$(2.1) \quad J = \begin{bmatrix} b_1 & a_1 & & \\ a_1 & b_2 & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & a_{n-1} & b_n \end{bmatrix}$$

with $a_j > 0$, $b_j \in \mathbb{R}$. An equivalent statement is that the orthonormal polynomials obey a three-term recurrence:

$$xp_j(x) = a_j p_{j+1}(x) + b_j p_j(x) + a_{j-1} p_{j-1}(x)$$

where $a_{-1} = 0$ and $p_n \equiv 0$. A third equivalent statement is the following: λ is an eigenvalue of J if and only if $\lambda \in \text{supp}(d\nu)$; moreover, the corresponding eigenvector is $[p_0(\lambda), p_1(\lambda), \dots, p_{n-1}(\lambda)]^T$.

We have just shown how measures on \mathbb{R} lead to Jacobi matrices; in fact, there is a one-to-one correspondence between them. Given a Jacobi matrix, J , let $d\nu$ be the spectral measure associated to J and the vector $e_1 = [1, 0, \dots, 0]^T$. Then J represents $x \mapsto xf(x)$ in the basis of orthonormal polynomials associated to $d\nu$.

Before explaining the origin of CMV matrices, it is necessary to delve a little into the theory of orthogonal polynomials on the unit circle. For a more complete description of what follows, the reader should turn to [Sim1]. Given a finitely-supported probability measure $d\mu$ on S^1 , the unit circle in \mathbb{C} , we can construct an orthonormal system of polynomials, ϕ_k , by applying the Gram–Schmidt procedure to $\{1, z, z^2, \dots\}$. These obey a recurrence relation; however, to simplify the formulae, we will present the relation for the monic orthogonal polynomials $\Phi_k(z)$:

$$(2.2) \quad \Phi_{k+1}(z) = z\Phi_k(z) - \bar{\alpha}_k \Phi_k^*(z).$$

Here α_k are recurrence coefficients, which are called Verblunsky coefficients, and Φ_k^* denotes the reversed polynomial:

$$(2.3) \quad \Phi_k(z) = \sum_{l=0}^k c_l z^l \quad \Rightarrow \quad \Phi_k^*(z) = \sum_{l=0}^k \bar{c}_{k-l} z^l = z^k \overline{\Phi_k(1/\bar{z})}.$$

When $d\mu$ is supported at exactly n points, $\alpha_k \in \mathbb{D}$ for $0 \leq k \leq n-2$ while α_{n-1} is a unimodular complex number. (Incidentally, if $d\mu$ has infinite support, then there are infinitely many Verblunsky coefficients and all lie inside the unit disk.) The Verblunsky coefficients completely describe the measure $d\mu$:

Theorem 1 (Verblunsky). *There is a 1-to-1 correspondence between probability measures on the unit circle supported at n points and Verblunsky coefficients $(\alpha_0, \dots, \alpha_{n-1})$ with $\alpha_k \in \mathbb{D}$ for $0 \leq k \leq n-2$ and $\alpha_{n-1} \in S^1$.*

From the discussion of Jacobi matrices, it would be natural to consider a matrix representation of $f(z) \mapsto zf(z)$ in $L^2(d\mu)$. Cantero, Moral, and Velázquez had the simple and ingenious idea to define a basis in $L^2(d\mu)$ by applying the Gram–Schmidt procedure to $\{1, z, z^{-1}, z^2, z^{-2}, \dots\}$. The resulting functions, $\chi_k(z)$ ($0 \leq k \leq n-1$), are easily expressed in terms of the orthonormal polynomials:

$$(2.4) \quad \chi_k(z) = \begin{cases} z^{-k/2} \phi_k^*(z) & : k \text{ even} \\ z^{-(k-1)/2} \phi_k(z) & : k \text{ odd.} \end{cases}$$

In the orthonormal basis $\{\chi_k(z)\}$ of $L^2(d\mu)$, the operator $f(z) \mapsto zf(z)$ is represented by the CMV matrix associated to the Verblunsky coefficients of the measure $d\mu$: Given the Verblunsky coefficients $\alpha_0, \dots, \alpha_{n-2}$ in \mathbb{D} and $\alpha_{n-1} \in S^1$ associated to the measure $d\mu$, let $\rho_k = \sqrt{1 - |\alpha_k|^2}$, and define 2×2 matrices

$$\Xi_k = \begin{bmatrix} \bar{\alpha}_k & \rho_k \\ \rho_k & -\alpha_k \end{bmatrix}$$

for $0 \leq k \leq n-2$, while $\Xi_{-1} = [1]$ and $\Xi_{n-1} = [\bar{\alpha}_{n-1}]$ are 1×1 matrices. From these, form the $n \times n$ block-diagonal matrices

$$\mathcal{L} = \text{diag}(\Xi_0, \Xi_2, \Xi_4, \dots) \quad \text{and} \quad \mathcal{M} = \text{diag}(\Xi_{-1}, \Xi_1, \Xi_3, \dots).$$

The CMV matrix associated to the coefficients $\alpha_0, \dots, \alpha_{n-1}$ is $\mathcal{C} = \mathcal{L}\mathcal{M}$.

The measure $d\mu$ can be reconstructed from \mathcal{C} in a manner analogous to the Jacobi case:

Theorem 2. *Let $d\mu$ be the spectral measure associated to a CMV matrix, \mathcal{C} , and the vector e_1 . Then \mathcal{C} is the CMV matrix associated to the measure $d\mu$.*

Proofs of these Theorems can be found in [CanMorVell] or [Sim1]. As explained in the Introduction, throughout the paper we will always use implicitly the bijection between measures $d\mu = \sum \delta_{z_j} \mu_j$, CMV matrices, and the coordinates given by the z_j 's and μ_j 's. A very important notion, that will be heavily used in this paper, is the *Carathéodory function* associated to a probability measure μ on the unit circle S^1 ; it is given by (see [Sim1, Section 1.3])

$$(2.5) \quad F(z) = \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta).$$

In terms of the other coordinates, F is given by

$$(2.6) \quad F(z) = \left(\frac{\mathcal{C} + z}{\mathcal{C} - z} \right)_{11} = \sum_{j=1}^n \frac{z_j + z}{z_j - z} \mu_j,$$

where \mathcal{C} is the CMV matrix associated to the measure μ . Moreover, the Carathéodory function is related to the *Schur function* f by

$$(2.7) \quad F(z) = \frac{1 + zf(z)}{1 - zf(z)} \quad \Longleftrightarrow \quad f(z) = \frac{1}{z} \frac{1 - F(z)}{1 + F(z)}.$$

While the Carathéodory function plays a very important role throughout the theory of orthogonal polynomials on the unit circle, it is very simple to see from the formulae above that, in the case of finite measures, F encodes exactly the same information as the measure μ . This is exactly the reason why we can use the Carathéodory function in a functional analytic approach to describe all the Poisson structures that we will introduce.

2.2. Integrable systems and classical R-matrices. The manifold of Jacobi matrices with fixed trace forms a co-adjoint orbit of the group of invertible upper triangular matrices, if one views the space of symmetric matrices as a dual space to the algebra of upper triangular matrices. The Lie-Poisson structure on this dual thus induces a symplectic structure on the Jacobi orbit. More generally, the space of 3-diagonal (not necessarily symmetric) matrices form a Poisson submanifold in $gl(n; \mathbb{R})$ with respect to the Lie-Poisson bracket associated to a particular Lie algebra structure on the $n \times n$ matrices (although not the one defined via the usual matrix commutator). These matters are described in detail in [Dei, OPRS, Per], for example. In contrast, CMV matrices are elements of the unitary group and hence the natural backdrop for CMV is that of Poisson-Lie groups or, more specifically, the group $Gl(n, \mathbb{C})$ equipped with the Sklyanin bracket (see [OPRS, §2.12]). However, we choose to give a presentation in which the associative algebra of $n \times n$ matrices takes center stage; an analogous construction for KdV using the algebra of pseudo-differential operators was given by Gelfand and Dikij [GelDik]. This approach is described in Section 2.12.6 of [OPRS]. (Note that here we are referring to the second symplectic structure associated with KdV, which was originally proposed by Adler [Adl, §4].)

Let \mathfrak{g} denote the (associative) algebra of $n \times n$ complex matrices. The algebra structure gives rise to a natural Lie algebra structure:

$$[B, C] = BC - CB.$$

As a vector space, $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{a}$, where

$$\mathfrak{a} = \{A : A = -A^\dagger\},$$

is the space of skew-Hermitian matrices, which is the Lie algebra of the group $\mathbb{U}(n)$ of $n \times n$ unitary matrices, and

$$\mathfrak{l} = \{A \in \mathfrak{g} : L_{i,j} = 0 \text{ for } i > j \text{ and } L_{i,i} \in \mathbb{R}\}$$

is the space of upper triangular matrices with real diagonal entries (the Lie algebra of the group $\mathbb{L}(n)$ of $n \times n$ lower triangular matrices with positive diagonal entries). We will write $\pi_{\mathfrak{a}}$ and $\pi_{\mathfrak{l}}$ for the natural projections onto these summands. This vector-space splitting of \mathfrak{g} gives rise to a second Lie algebra structure. First we define $R : \mathfrak{g} \rightarrow \mathfrak{g}$ by either

$$(2.8) \quad \begin{aligned} R(X) &= \pi_{\mathfrak{l}}(X) - \pi_{\mathfrak{a}}(X), & \text{for all } X \in \mathfrak{g}, \text{ or} \\ R(L + A) &= L - A & \text{for all } A \in \mathfrak{a} \text{ and } L \in \mathfrak{l}. \end{aligned}$$

The second Lie bracket can then be written as either

$$(2.9) \quad \begin{aligned} [X, Y]_R &= \frac{1}{2}[R(X), Y] + \frac{1}{2}[X, R(Y)] \quad \forall X, Y \in \mathfrak{g}, \text{ or} \\ [L + A, L' + A']_R &= [L, L'] - [A, A'] \quad \forall L, L' \in \mathfrak{l}, \text{ and } A, A' \in \mathfrak{a}. \end{aligned}$$

The second definition also makes it transparent that the bracket $[\cdot, \cdot]_R$ obeys the Jacobi identity. But the main property that we are interested in is that R obeys the modified classical Yang-Baxter equation:

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = -[X, Y]$$

This allows us to define Poisson brackets as follows: We can identify the dual space \mathfrak{g}^* with \mathfrak{g} using the pairing

$$(2.10) \quad \langle X, Y \rangle = \text{Im Tr}(XY).$$

The form $\langle \cdot, \cdot \rangle$ is non-degenerate symmetric and *invariant*:

$$(2.11) \quad \langle X, [Z, Y] \rangle = \text{Im Tr}(XZY - XYZ) = \langle [X, Z], Y \rangle$$

or, equivalently,

$$(2.12) \quad \langle BXB^{-1}, BYB^{-1} \rangle = \langle X, Y \rangle, \quad \text{for any } B \in GL(n, \mathbb{C}).$$

Given a smooth function $\varphi : \mathfrak{g} \rightarrow \mathbb{R}$ and $B \in \mathfrak{g}$, define $\nabla\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$ by

$$(2.13) \quad \left. \frac{d}{dt} \right|_{t=0} \varphi(B + tC) = \langle \nabla\varphi|_B, C \rangle.$$

Equivalently, if we write $b_{k,l} = u_{k,l} + iv_{k,l}$ for the matrix entries of B , then

$$(2.14) \quad [\nabla\varphi]_{k,l} = \frac{\partial\varphi}{\partial v_{l,k}} + i \frac{\partial\varphi}{\partial u_{l,k}}.$$

We can now define the desired Poisson brackets on \mathfrak{g} . Let $R : \mathfrak{g} \rightarrow \mathfrak{g}$ and $\langle \cdot, \cdot \rangle$ be as above. Given $\varphi_1, \varphi_2 : \mathfrak{g} \rightarrow \mathbb{R}$, let $\nabla_j = \nabla\varphi_j$ for $j = 1, 2$. Then both

$$(2.15) \quad \{\varphi_1, \varphi_2\}_{LP}|_X = \frac{1}{2}\langle [\nabla_1, \nabla_2]_R, X \rangle$$

and

$$(2.16) \quad \{\varphi_1, \varphi_2\}_{GD}|_X = \frac{1}{2}\langle R(\nabla_1 X), \nabla_2 X \rangle - \frac{1}{2}\langle R(X\nabla_1), X\nabla_2 \rangle$$

define Poisson structures on \mathfrak{g} , the first one known as the Lie-Poisson (LP) bracket, and the second known as the Gelfand-Dikij (GD) bracket. Then under the embedding $J \mapsto iJ$, the manifolds of Jacobi matrices with fixed trace are symplectic leaves in $(\mathfrak{g}, \{\cdot, \cdot\}_{LP})$; this is just the usual construction in $GL(n, \mathbb{R})$ in disguise. Similarly, the manifold of CMV matrices with fixed determinant forms a symplectic leaf in the Poisson manifold $(\mathfrak{g}, \{\cdot, \cdot\}_{GD})$. Furthermore, the

restriction of the GD Poisson structure to the manifold of CMV matrices coincides with the AL Poisson structure (1.3) (see [KilNen2]).

3. POISSON BRACKETS FOR CARATHÉODORY AND SCHUR FUNCTIONS

Let ∇_1 and ∇_2 be the gradients at a point X of two functions φ_1 and φ_2 , as defined in Subsection 2.2. Then a simple calculation shows that the two brackets defined above can be written in the following slightly modified form:

$$(3.1) \quad \{\varphi_1, \varphi_2\}_{LP} = \frac{1}{4}\langle [X, \nabla_1], R(2\nabla_2) \rangle - \frac{1}{4}\langle [X, \nabla_2], R(2\nabla_1) \rangle$$

and

$$(3.2) \quad \{\varphi_1, \varphi_2\}_{GD} = \frac{1}{4}\langle [X, \nabla_1], R(X\nabla_2 + \nabla_2 X) \rangle - \frac{1}{4}\langle [X, \nabla_2], R(X\nabla_1 + \nabla_1 X) \rangle.$$

Before going any further, we give the following definition. Consider $k \geq 0$ and two smooth functions φ_1 and φ_2 , with gradients ∇_1 and ∇_2 respectively. Then we set

$$(3.3) \quad 4\{\varphi_1, \varphi_2\}^{(k)} = \left\langle X, [\nabla_1, R(X^k \nabla_2 + \nabla_2 X^k)] + [R(X^k \nabla_1 + \nabla_1 X^k), \nabla_2] \right\rangle.$$

One must note that this expression is a Poisson bracket only for $k = 0$, in which case it is exactly the Lie-Poisson bracket, and for $k = 1$. In this latter case a simple calculation using the properties of R and (3.2) shows that $\{\cdot, \cdot\}^{(1)}$ is actually the Gelfand-Dikij bracket defined in (2.16). But even though (3.3) is not in general a Poisson bracket, in what follows we will work with the general expression as it makes it easier to emphasize the relevant steps of our calculations.

We now focus on functions given by

$$\varphi(X) = \text{Im Tr}(P\Phi(X)P) = \text{Im}(\Phi(X)_{11}),$$

where Φ is a smooth function on $M_n(\mathbb{C})$, and P denotes the orthogonal projection on the vector $e_1 = (1, 0, \dots, 0)^T$,

$$P = e_1^T e_1.$$

Note that any φ defined this way is invariant under the conjugation by invertible matrices with off-diagonal entries in the first column and row all equal to zero:

$$\varphi(CXC^{-1}) = \varphi(X)$$

for

$$C = \begin{pmatrix} c_1 & 0 \\ 0 & C_2 \end{pmatrix},$$

where $c_1 \neq 0$ and C_2 is an invertible $(n-1) \times (n-1)$ matrix.

Lemma 3.1. *Let $\varphi(X) = \text{Im}(\Phi(X)_{11})$ as above. Then for any matrix $B \in M_n(\mathbb{C})$ we have*

$$(3.4) \quad \langle \nabla \varphi|_X, [X, B] \rangle = \langle \nabla \varphi|_X, [X, BP + PB] \rangle = \langle [\nabla \varphi|_X, X], BP + PB \rangle$$

where, as above, $P = e_1^T e_1$ is the orthogonal projection on the vector $e_1 = (1, 0, \dots, 0)^T$.

Proof. Let $C = \exp(tA)$, where A is any matrix of the same form as C . Then

$$0 = \left. \frac{d}{dt} \right|_{t=0} \varphi(CXC^{-1}) = \langle \nabla \varphi|_X, [A, X] \rangle,$$

which implies that, for any B an expression $\langle \nabla \varphi|_X, [X, B] \rangle$ does not depend on the $(n-1) \times (n-1)$ submatrix of B obtained by deleting the first row and column. \square

Using the properties of the R -matrix and Lemma 3.1 we see that:

$$\begin{aligned}
4\{\varphi_1, \varphi_2\}^{(k)} &= \left\langle [X, \nabla_1], R(X^k \nabla_2 + \nabla_2 X^k) \right\rangle - \left\langle [X, \nabla_2], R(X^k \nabla_1 + \nabla_1 X^k) \right\rangle \\
&= \left\langle \nabla_1, [R(X^k \nabla_2 + \nabla_2 X^k), X] \right\rangle - \left\langle \nabla_2, [R(X^k \nabla_1 + \nabla_1 X^k), X] \right\rangle \\
&= \left\langle \nabla_1, [PR(X^k \nabla_2 + \nabla_2 X^k) + R(X^k \nabla_2 + \nabla_2 X^k)P, X] \right\rangle \\
&\quad - \left\langle \nabla_2, [PR(X^k \nabla_1 + \nabla_1 X^k) + R(X^k \nabla_1 + \nabla_1 X^k)P, X] \right\rangle \\
&= \left\langle [X, \nabla_1], PR(X^k \nabla_2 + \nabla_2 X^k) + R(X^k \nabla_2 + \nabla_2 X^k)P \right\rangle \\
&\quad - \left\langle [X, \nabla_2], PR(X^k \nabla_1 + \nabla_1 X^k) + R(X^k \nabla_1 + \nabla_1 X^k)P \right\rangle
\end{aligned}$$

While this expression appears to be more complicated than the one we started from, it will shortly be shown that it is exactly what we need in order to continue our calculation.

Using Lemma 3.1, we can compute the Poisson bracket for functions φ as above. Note that this calculation is general and extremely robust, and it applies to a large range of expressions involving R -matrices. The first such calculation that we are aware of was done for the finite Toda lattice by Faybusovich and Gekhtman [FayGek2]. Proposition 3.2 gives a short proof of the next step in the calculation, while Theorem 3 takes it to its conclusion.

Proposition 3.2. *Let $\varphi_j(X) = \text{Im}(\Phi_j(X)_{11}) = \text{Im} \text{Tr}(\Phi_j(X)P)$ for $j = 1, 2$, where Φ_j are smooth functions, and set $\nabla_j = \nabla \varphi_j(X)$. Then the values of the linear and quadratic brackets of the φ_j 's are given by:*

$$(3.5) \quad 2\{\varphi_1, \varphi_2\}_{LP}|_X = \text{Im} \left(2(X \nabla_1 \nabla_2^*)_{11} - 2(X \nabla_2 \nabla_1^*)_{11} + [X, [\nabla_1, \nabla_2]]_{11} \right)$$

and

$$\begin{aligned}
(3.6) \quad 2\{\varphi_1, \varphi_2\}_{GD}|_X &= \text{Im} \left(2(X \nabla_1 \nabla_2^* X^*)_{11} - 2(\nabla_1 X X^* \nabla_2^*)_{11} \right. \\
&\quad \left. + [X \nabla_1, X \nabla_2]_{11} - [\nabla_1 X, \nabla_2 X]_{11} \right).
\end{aligned}$$

Proof. First note that for any matrix A we have

$$PR(A) = 2 \text{Re}(A_{11})P - PA \quad \text{and} \quad R(A)P = AP + 2A^*P - 2 \text{Re}(A_{11})P.$$

Therefore

$$PR(A) + R(A)P = [A, P] + 2A^*P$$

and so for any $k \geq 0$ the expression $4\{\varphi_1, \varphi_2\}^{(k)}$ equals

$$\begin{aligned}
(3.7) \quad &\left\langle [X, \nabla_1], 2(X^k \nabla_2 + \nabla_2 X^k)^* P \right\rangle - \left\langle [X, \nabla_2], 2(X^k \nabla_1 + \nabla_1 X^k)^* P \right\rangle \\
&+ \left\langle [X, \nabla_1], [X^k \nabla_2 + \nabla_2 X^k, P] \right\rangle - \left\langle [X, \nabla_2], [X^k \nabla_1 + \nabla_1 X^k, P] \right\rangle
\end{aligned}$$

Taking (3.7) for $k = 0$ and 1 yields the result. \square

Note that the R -matrix does not appear any more in (3.5) or (3.6). This allows us to prove our first main result: compute the respective brackets for the Weyl and Carathéodory functions. More precisely, let $X \in M_n(\mathbb{C})$ be a matrix, and $z, \lambda \in \mathbb{C} \setminus \text{spec}(X)$ complex numbers. For $k \geq 0$, consider

$$(3.8) \quad m^{(k)}(\lambda) = (X^k(\lambda - X)^{-1})_{11} = \text{Tr}(X^k(\lambda - X)^{-1}P)$$

and

$$(3.9) \quad F^{(k)}(z) = (X^k(X + z)(X - z)^{-1})_{11} = \text{Tr}(X^k(X + z)(X - z)^{-1}P).$$

We think of these functions as being defined on the space of matrices, and depending on a complex parameter, λ or z respectively. Note that, for $k = 0$, we recover the Weyl and Carathéodory functions, respectively. We then have the following result:

Theorem 3. Let $X \in M_n(\mathbb{C})$ be a fixed matrix, and $z, w, \lambda, \xi \in \mathbb{C} \setminus \text{spec}(X)$ be complex parameters, with $z \neq w$ and $\lambda \neq \xi$. Then, for $k \geq 0$,

$$(3.10) \quad \{m(\lambda), m(\xi)\}^{(k)} = (m^{(k)}(\lambda) - m^{(k)}(\xi)) \left[m(\lambda)m(\xi) - \frac{m(\lambda) - m(\xi)}{\lambda - \xi} \right]$$

and

$$(3.11) \quad \{F(z), F(w)\}^{(k+1)} = i(F^{(k)}(z) - F^{(k)}(w)) \left[F(z)F(w) - 1 - \frac{z+w}{z-w}(F(z) - F(w)) \right].$$

Corollary 3.3. Let \mathcal{C} be a finite CMV matrix, and F and f the Carathéodory and Schur functions associated to the spectral measure of \mathcal{C} and $e_1 = [1, 0, \dots, 0]^T$. Then, for two distinct points $z, w \in \mathbb{C} \setminus \text{spec}(\mathcal{C})$, the GD brackets of these functions at \mathcal{C} are given by:

$$(3.12) \quad \begin{aligned} \{F(z), F(w)\}_{GD} &= i(F(z) - F(w))(F(z)F(w) - 1) \\ &\quad - i \frac{z+w}{z-w} (F(z) - F(w))^2 \end{aligned}$$

and

$$(3.13) \quad \{f(z), f(w)\}_{GD} = -2i \frac{f(z) - f(w)}{z - w} (zf(z) - wf(w)).$$

Proof. The first relation is just a special case of (3.11) for $k = 0$ and $X = \mathcal{C}$. The bracket (3.13) follows from (2.7) and the observation that

$$\begin{aligned} \{f(z), f(w)\}_{GD} &= \frac{1}{zw} \frac{d}{dF(z)} \left(\frac{1 - F(z)}{1 + F(z)} \right) \frac{d}{dF(w)} \left(\frac{1 - F(w)}{1 + F(w)} \right) \{F(z), F(w)\}_{GD} \\ &= \frac{4}{zw} \frac{1}{(1 + F(z))^2 (1 + F(w))^2} \{F(z), F(w)\}_{GD}. \end{aligned}$$

Now use (3.12) and the expression (2.7) of F in terms of f . □

Proof of Theorem 3. The proofs of both relations follow the exact same ideas, but since in this paper we focus on the Ablowitz-Ladik system, and hence the Gelfand-Dikij bracket, we will only give the proof of (3.11). We approach this by first computing the k -brackets of $\text{Im } F$ and $\text{Re } F = \text{Im}(iF)$ at two different points z and w .

Let R be the R-matrix defined in Section 2. We start by working with the general expression for the k -bracket and for any two functions φ_1 and φ_2 as above. We know that

$$\begin{aligned} 4\{\varphi_1, \varphi_2\}^{(k)}(X) &= \left\langle [X, \nabla_1], PR(X^k \nabla_2 + \nabla_2 X^k) + R(X^k \nabla_2 + \nabla_2 X^k)P \right\rangle \\ &\quad - \left\langle [X, \nabla_2], PR(X^k \nabla_1 + \nabla_1 X^k) + R(X^k \nabla_1 + \nabla_1 X^k)P \right\rangle \end{aligned}$$

Just as in the proof of Proposition 3.2, the observation that allows us to continue is that, for any matrix A :

$$PR(A) = 2 \text{Re}(A_{11})P - PA \quad \text{and} \quad R(A)P = AP + 2A^*P - 2 \text{Re}(A_{11})P.$$

Therefore

$$PR(A) + R(A)P = [A, P] + 2A^*P$$

and so

$$(3.14) \quad \begin{aligned} 4\{\varphi_1, \varphi_2\}^{(k)}(X) &= \left\langle [X, \nabla_1], 2(X^k \nabla_2 + \nabla_2 X^k)^* P \right\rangle \\ &\quad - \left\langle [X, \nabla_2], 2(X^k \nabla_1 + \nabla_1 X^k)^* P \right\rangle \\ &\quad + \left\langle [X, \nabla_1], [X^k \nabla_2 + \nabla_2 X^k, P] \right\rangle \\ &\quad - \left\langle [X, \nabla_2], [X^k \nabla_1 + \nabla_1 X^k, P] \right\rangle \end{aligned}$$

We wish to apply this formula to the real and imaginary parts of the Carathéodory function F

$$F(z) = \left(\frac{X+z}{X-z} \right)_{11}.$$

Indeed, let $u(z) = \operatorname{Re} F(z) = \operatorname{Im} iF(z)$ and $v(z) = \operatorname{Im} F(z)$. Note that, for a fixed parameter z , these functions are well-defined in a neighborhood of X , and they are of the type we have considered above. If we denote $\nabla_z = \nabla v(z)$, then we get

$$\nabla u(z) = i\nabla_z = -2iz(X-z)^{-1}P(X-z)^{-1}$$

and

$$\nabla v(z) = \nabla_z = -2z(X-z)^{-1}P(X-z)^{-1}.$$

Plugging these into the formula for the k^{th} R -bracket, we get

$$\begin{aligned} \{u(z), u(w)\}^{(k)} &= \operatorname{Im}(T_1(k) - T_2(k) - T_3(k) + T_4(k)) \\ \{v(z), v(w)\}^{(k)} &= \operatorname{Im}(T_1(k) - T_2(k) + T_3(k) - T_4(k)) \\ \{u(z), v(w)\}^{(k)} &= \operatorname{Im}(iT_1(k) + iT_2(k) + iT_3(k) - iT_4(k)) \\ &= \operatorname{Re}(T_1(k) + T_2(k) + T_3(k) - T_4(k)) \\ \{v(z), u(w)\}^{(k)} &= \operatorname{Im}(-iT_1(k) - iT_2(k) + iT_3(k) - iT_4(k)) \\ &= \operatorname{Re}(-T_1(k) - T_2(k) + T_3(k) - T_4(k)), \end{aligned}$$

where

$$\begin{aligned} T_1(k) &= \frac{1}{4} \operatorname{Tr} \left(2[X, \nabla_z] (X^k \nabla_w + \nabla_w X^k)^* P \right) \\ T_2(k) &= \frac{1}{4} \operatorname{Tr} \left(2[X, \nabla_w] (X^k \nabla_z + \nabla_z X^k)^* P \right) \\ T_3(k) &= \frac{1}{4} \operatorname{Tr} \left([X, \nabla_z] \cdot [X^k \nabla_w + \nabla_w X^k, P] \right) \\ T_4(k) &= \frac{1}{4} \operatorname{Tr} \left([X, \nabla_w] \cdot [X^k \nabla_z + \nabla_z X^k, P] \right). \end{aligned}$$

Therefore

$$\begin{aligned} \{F(z), F(w)\}^{(k)} &= \{u(z), u(w)\}^{(k)} - \{v(z), v(w)\}^{(k)} \\ &\quad + i(\{u(z), v(w)\}^{(k)} + \{v(z), u(w)\}^{(k)}) \\ &= -2 \operatorname{Im}(T_3(k) - T_4(k)) + i \cdot 2 \operatorname{Re}(T_3(k) - T_4(k)) \\ &= 2i(T_3(k) - T_4(k)) \end{aligned}$$

Let us note in passing that in the $k=1$ (Gelfand-Dikij) case, we have that $T_1 = T_3$ and $T_2 = T_4$.

In order to compute $T_3 - T_4$, we also need the following simple observation: For any three matrices A, B and C ,

$$(3.15) \quad \operatorname{Tr}(APBPCP) = \operatorname{Tr}(PAPBPCP) = A_{11}B_{11}C_{11}.$$

After simplifying and grouping terms together, we get that

$$4T_3(k) - 4T_4(k) = \text{I}(k) + \text{II}(k) + \text{III}(k),$$

where

$$\begin{aligned} \text{I}(k) &= 8zw((X-z)^{-1}X^k(X-w)^{-1})_{11} \cdot \left[(X(X-z)^{-1})_{11}((X-w)^{-1})_{11} \right. \\ &\quad \left. - (X(X-w)^{-1})_{11}((X-z)^{-1})_{11} \right], \\ \text{II}(k) &= 8zw((X-z)^{-1}X(X-w)^{-1})_{11} \cdot \left[(X^k(X-z)^{-1})_{11}((X-w)^{-1})_{11} \right. \\ &\quad \left. - (X^k(X-w)^{-1})_{11}((X-z)^{-1})_{11} \right], \end{aligned}$$

and

$$\text{III}(k) = 8zw((X-z)^{-1}(X-w)^{-1})_{11} \cdot \left[(X(X-z)^{-1})_{11}(X^k(X-w)^{-1})_{11} - (X(X-w)^{-1})_{11}(X^k(X-z)^{-1})_{11} \right].$$

Recall that, for $j \geq 0$, we have defined the function

$$(3.16) \quad F^{(j)}(z) = \left(X^j \frac{X+z}{X-z} \right)_{11}, \quad j \in \mathbb{Z}.$$

Then we have that, for $j \in \mathbb{Z}$,

$$(3.17) \quad (X^{j+1}(X-z)^{-1})_{11} = \frac{1}{2}(F^{(j)}(z) + F^{(j)}(0)) = \frac{1}{2z}(F^{(j+1)}(z) - F^{(j+1)}(0)),$$

$$(3.18) \quad ((X-z)^{-1}X^{j+1}(X-w)^{-1})_{11} = \frac{1}{2(z-w)}(F^{(j)}(z) - F^{(j)}(w)),$$

and

$$(3.19) \quad ((X-z)^{-1}(X-w)^{-1})_{11} = \frac{1}{2(z-w)} \left(\frac{1}{z}(F(z) - 1) - \frac{1}{w}(F(w) - 1) \right).$$

We will use the appropriate formula of (3.17)–(3.19) in order to express our quantities only in terms of $F \equiv F^{(0)}$ and $F^{(k)}$. Thus we get

$$(3.20) \quad (z-w) \cdot \text{I}(k+1) = (F^{(k)}(z) - F^{(k)}(w)) [z(F(z) + 1)(F(w) - 1) - w(F(w) + 1)(F(z) - 1)],$$

$$(3.21) \quad (z-w) \cdot \text{II}(k+1) = (F(z) - F(w)) [z(F^{(k)}(z) + F^{(k)}(0))(F(w) - 1) - w(F^{(k)}(w) + F^{(k)}(0))(F(z) - 1)],$$

and

$$(3.22) \quad (z-w) \cdot \text{III}(k+1) = (w(F(z) - 1) - z(F(w) - 1)) \times [(F(z) + 1)(F^{(k)}(w) + F^{(k)}(0)) - (F(w) + 1)(F^{(k)}(z) + F^{(k)}(0))].$$

A straightforward calculation shows that $\text{II} + \text{III} = \text{I}$, and hence we find that

$$(3.23) \quad \{F(z), F(w)\}^{(k+1)} = i(F^{(k)}(z) - F^{(k)}(w)) \left[F(z)F(w) - 1 - \frac{z+w}{z-w}(F(z) - F(w)) \right],$$

as claimed. \square

We wish to use (3.12) to find the bracket of the eigenvalues and masses of the spectral measure, $\mu = \sum_{j=1}^n \delta_{z_j} \mu_j$, $z_j = e^{i\theta_j}$ and $\sum \mu_j = 1$, of a CMV matrix \mathcal{C} . Note that, since for the next couple of sections we only work with the GD bracket, we will not specify it in order to simplify notation. We hope that this will not cause any confusion.

Fix a CMV matrix \mathcal{C} , with $d\mu = \sum_{j=1}^n \delta_{z_j} \mu_j$, $z_j = e^{i\theta_j}$ and $\sum \mu_j = 1$, the associated spectral measure. Let us expand the bracket $\{F(z), F(w)\}|_{\mathcal{C}}$ in terms of the z_j s and μ_j s:

$$\begin{aligned} \{F(z), F(w)\} &= \sum_{j,k=1}^n \left\{ \frac{z+z_j}{z-z_j} \mu_j, \frac{w+z_k}{w-z_k} \mu_k \right\} \\ &= \sum_{j,k=1}^n \frac{z+z_j}{z-z_j} \frac{w+z_k}{w-z_k} \{\mu_j, \mu_k\} \\ &\quad + \sum_{j,k=1}^n \mu_j \frac{w+z_k}{w-z_k} \frac{d}{dz_j} \left(\frac{z+z_j}{z-z_j} \right) \{z_j, \mu_k\} \\ &\quad + \sum_{j,k=1}^n \mu_j \mu_k \frac{d}{dz_j} \left(\frac{z+z_j}{z-z_j} \right) \frac{d}{dz_k} \left(\frac{w+z_k}{w-z_k} \right) \{z_j, z_k\}. \end{aligned}$$

Since

$$\frac{d}{d\zeta} \left(\frac{z + \zeta}{z - \zeta} \right) = \frac{2z}{(z - \zeta)^2},$$

we obtain

$$\begin{aligned} \{F(z), F(w)\} &= \sum_{j,k=1}^n \frac{(z + z_j)(w + z_k)}{(z - z_j)(w - z_k)} \cdot \{\mu_j, \mu_k\} \\ &\quad + \sum_{j,k=1}^n \frac{2z(w + z_k)\mu_j}{(z - z_j)^2(w - z_k)} \cdot \{z_j, \mu_k\} \\ &\quad + \sum_{j,k=1}^n \frac{4zw\mu_j\mu_k}{(z - z_j)^2(w - z_k)^2} \cdot \{z_j, z_k\}. \end{aligned}$$

Let $1 \leq s, t \leq n$. If $s \neq t$, then we choose Γ_s and Γ_t to be small, positively oriented contours around z_s and z_t , respectively; we require that they do not intersect, nor surround more than one eigenvalue. In the case $s = t$, we choose two small, positively oriented contours Γ_s and Γ'_s around z_s so that the contour Γ_s is completely contained in the interior of Γ'_s . By the residue formula the previous expansion implies that

$$(3.24) \quad \frac{1}{(2\pi i)^2} \int_{\Gamma_t} \int_{\Gamma_s} \{F(z), F(w)\} dz dw = 4z_s z_t \{\mu_s, \mu_t\},$$

$$(3.25) \quad \frac{1}{(2\pi i)^2} \int_{\Gamma_t} \int_{\Gamma_s} (z - z_s) \{F(z), F(w)\} dz dw = 4z_s z_t \mu_s \{z_s, \mu_t\},$$

and

$$(3.26) \quad \frac{1}{(2\pi i)^2} \int_{\Gamma_t} \int_{\Gamma_s} (z - z_s)(w - z_t) \{F(z), F(w)\} dz dw = 4z_s z_t \mu_s \mu_t \{z_s, z_t\}.$$

Note that, if $s = t$, then we set $\Gamma_t = \Gamma'_s$; in other words, we first integrate over the smaller of the two contours around z_s .

Theorem 4. *With the definitions from the previous sections we have that, in the GD Poisson structure,*

$$(3.27) \quad \{z_s, z_t\} = 0,$$

$$(3.28) \quad \{z_s, \mu_t\} = 2iz_s \mu_t (\delta_{st} - \mu_s),$$

and

$$(3.29) \quad \{\mu_s, \mu_t\} = 2i\mu_s \mu_t \left[\sum_{k \neq s} \frac{z_k + z_s}{z_k - z_s} \mu_k + \sum_{k \neq t} \frac{z_t + z_k}{z_t - z_k} \mu_k + \frac{z_s + z_t}{z_s - z_t} \right].$$

Proof. We prove the result by finding the residues generated by the right-hand side of

$$(3.30) \quad \begin{aligned} \{F(z), F(w)\} &= i(F(z) - F(w))(F(z)F(w) - 1) \\ &\quad - i \frac{z + w}{z - w} (F(z) - F(w))^2 \end{aligned}$$

in the integrals from (3.24), (3.25), and (3.26).

We begin with (3.26). Note that only the quadratic poles in both $z - z_j$ and $w - z_k$ play any role. But for $s \neq t$ there is no term on the right-hand side of (3.30) which contains the denominator $(z - z_s)^2(w - z_t)^2$, and hence the double integral over Γ_t and Γ_s is identically 0. This proves (3.27).

Now we turn to (3.25). Assume first that $s \neq t$. By the same reasoning than above, the only terms in (3.30) which contribute to the integral are the ones containing the denominator

$(z - z_s)^2(w - z_t)$. In this case, that translates into

$$\begin{aligned} 4z_s z_t \mu_s \{z_s, \mu_t\} &= \frac{1}{(2\pi i)^2} \int_{\Gamma_t} \int_{\Gamma_s} (z - z_s) i F(z)^2 F(w) dz dw \\ &= \frac{1}{(2\pi i)^2} \int_{\Gamma_t} \int_{\Gamma_s} i(z - z_s) \frac{(z + z_s)^2}{(z - z_s)^2} \mu_s^2 \frac{w + z_t}{z_t - w} \mu_t dz dw \\ &= (-i) 4z_s^2 \mu_s^2 2z_t \mu_t, \end{aligned}$$

or, equivalently,

$$(3.31) \quad \{z_s, \mu_t\} = -2iz_s \mu_s \mu_t$$

for $s \neq t$. The case $s = t$ is slightly more complicated because the factor $(z + w)/(z - w)$ in the second term of the right-hand side of (3.30) plays a role. Indeed,

$$\frac{1}{2\pi i} \int_{\Gamma_s} (z - z_s) \{F(z), F(w)\} dz = i 4z_s^2 \mu_s^2 F(w) - i \frac{z_s + w}{z_s - w} 4z_s^2 \mu_s^2.$$

Recall that Γ_s is contained in the interior of Γ'_s , and hence the function $z \rightarrow \frac{z+w}{z-w}$ is analytic on an open neighborhood of the interior of Γ_s as long as w is on Γ'_s . Now integrate on Γ'_s :

$$4z_s^2 \mu_s \{z_s, \mu_s\} = -8iz_s^2 \mu_s^2 z_s \mu_s - i(-2z_s) 4z_s^2 \mu_s^2 = 4z_s^2 \mu_s (-2iz_s \mu_s^2 + 2iz_s \mu_s).$$

Combining this with (3.31) gives (3.28).

Finally, we turn to (3.24). Here the formulae are more involved and all the terms play a role.

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma_s} \{F(z), F(w)\} dz &= -2iz_s \mu_s \left(F(w) \sum_{k \neq s} \frac{z_s + z_k}{z_s - z_k} \mu_k - 1 \right) \\ &\quad - 2iz_s \mu_s F(w) \left(\sum_{k \neq s} \frac{z_s + z_k}{z_s - z_k} \mu_k - F(w) \right) \\ &\quad - 4iz_s \mu_s \frac{z_s + w}{z_s - w} \left(\sum_{k \neq s} \frac{z_s + z_k}{z_s - z_k} \mu_k - F(w) \right). \end{aligned}$$

Integrating this on Γ_t for $t \neq s$ yields

$$\begin{aligned} 4z_s z_t \{\mu_s, \mu_t\} &= -8iz_s z_t \mu_s \mu_t \sum_{k \neq s} \frac{z_s + z_k}{z_s - z_k} \mu_k \\ &\quad + 8iz_s z_t \mu_s \mu_t \sum_{k \neq t} \frac{z_t + z_k}{z_t - z_k} \mu_k \\ &\quad + 8iz_s z_t \mu_s \mu_t \frac{z_s + z_t}{z_s - z_t}, \end{aligned}$$

or, after simplifications,

$$\{\mu_s, \mu_t\} = 2\mu_s \mu_t \left[- \sum_{k \neq s} i \frac{z_s + z_k}{z_s - z_k} \mu_k + \sum_{k \neq t} i \frac{z_t + z_k}{z_t - z_k} \mu_k + i \frac{z_s + z_t}{z_s - z_t} \right],$$

as claimed. \square

Remark 3.4. As an immediate consequence of Theorem 4 we can recover the analogous formulae for the Ablowitz-Ladik system obtained by Killip and Nenciu, [KilNen2]. Indeed, let us rewrite the brackets in terms of θ_s (where $z_s = e^{i\theta_s}$) and $\log[\mu_t]$. We easily obtain from (3.27) and (3.28) that

$$\{\theta_s, \theta_t\} = 0$$

and

$$\{\theta_s, \log[\mu_t]\} = 2\delta_{st} - 2\mu_s.$$

The first formula is a direct consequence of the Lax pairs in [KilNen2, Proposition 4.5], while the second one coincides with equation (72) from the same paper. Consider (3.29) and use the fact that

$$i \frac{e^{i\varphi_1} + e^{i\varphi_2}}{e^{i\varphi_1} - e^{i\varphi_2}} = \cot\left(\frac{\varphi_1 - \varphi_2}{2}\right).$$

Then we get that

$$\begin{aligned} \{\log[\mu_s], \log[\mu_t]\} &= \sum_{k \neq s} \cot\left(\frac{\theta_k - \theta_s}{2}\right) \mu_k + \sum_{k \neq t} \cot\left(\frac{\theta_t - \theta_k}{2}\right) \mu_k + \cot\left(\frac{\theta_s - \theta_t}{2}\right) \\ &= \sum_{k \neq s, t} \left[\cot\left(\frac{\theta_k - \theta_s}{2}\right) + \cot\left(\frac{\theta_t - \theta_k}{2}\right) + \cot\left(\frac{\theta_s - \theta_t}{2}\right) \right] \mu_k. \end{aligned}$$

In the last identity we use the fact that

$$1 - \mu_s - \mu_t = \sum_{k \neq s, t} \mu_k.$$

Note that we recovered Proposition 8.5 from [KilNen2].

While our proof of (3.27) and (3.28) is not necessarily shorter than the one from [KilNen2], the case of (3.29) is completely different: not only is this derivation much simpler and shorter, it also only uses the R -matrix formulation of the Gelfand-Dikij bracket. The approach of Killip and Nenciu exploits the asymptotics of the spectral parameters, and the expression of the bracket in terms of Verblunsky coefficients.

4. THE EXTENDED BRACKET FOR UNNORMALIZED MEASURES

We will now extend the bracket from probability measures μ on the unit circle S^1 to general, finite measures $\tilde{\mu}$ on S^1 . If $c = \tilde{\mu}(S^1)$ is the total weight, and F the Carathéodory function associated to the normalized measure, note that c and F fully characterize the unnormalized measure. In other words, in order to define the extended bracket it is sufficient to give its value for these two quantities:

$$(4.1) \quad \{c, F(z)\} = ic(F(z)^2 - 1),$$

and then extend it to all smooth functions using the Leibnitz rule and bi-linearity.³ While this describes the bracket uniquely, it is not clear that it obeys the Jacobi identity. In order to show this, we will rewrite the bracket in a different set of coordinates. Indeed, from (4.1) we get, by the same methods in the proof of Theorem 4, that

$$\{c, z_s\} = -2icz_s\mu_s \quad \text{and} \quad \{c, \mu_s\} = -2ic\mu_s \sum_{j \neq s} \frac{z_s + z_j}{z_s - z_j},$$

where for a finite measure $\tilde{\mu}$ we always denote by μ the associated probability measure, $\mu = \tilde{\mu}/|\tilde{\mu}|$ and, as in the previous sections, $\mu = \sum_{j=1}^n \mu_j \delta_{z_j}$, $z_j = e^{i\theta_j}$. Further set $\tilde{\mu}_j = c\mu_j$ to be the corresponding weights for the unnormalized measure $\tilde{\mu}$.⁴ Note that the space of measures $\tilde{\mu}$ is $2n$ -dimensional and parameterized by $\theta_1, \dots, \theta_n, \tilde{\mu}_1, \dots, \tilde{\mu}_n$. In these variables, a direct calculation using (3.27)-(3.29) and (4.1) shows that for all $1 \leq s, t \leq n$ the bracket is given by

$$(4.2) \quad \{\theta_s, \theta_t\} = 0, \quad \{\theta_s, \tilde{\mu}_t\} = 2\tilde{\mu}_t \delta_{st},$$

and, for $s \neq t$,

$$(4.3) \quad \{\tilde{\mu}_s, \tilde{\mu}_t\} = 2i\tilde{\mu}_s\tilde{\mu}_t \frac{z_s + z_t}{z_s - z_t} = 2\tilde{\mu}_s\tilde{\mu}_t \cot\left(\frac{\theta_s - \theta_t}{2}\right).$$

So now we can easily prove that

³Note that here we use the same notation for the Poisson bracket on the space of finite measures as for the one on the space of probability measures. But since the later is just the restriction of the former, we trust that this will not create any confusion.

⁴Recall that any symbol with a tilde refers to the unnormalized measures.

Proposition 4.1. *The bracket given by (4.2) and (4.3) on the space of finite measures supported at n points on the unit circle obeys the Jacobi identity, and hence it defines a Poisson bracket. Furthermore, this bracket is nondegenerate.*

Proof. Checking the Jacobi identity is actually a simple calculation if one uses the following (slight) variations of (4.2) and (4.3) obtained if we replace the $\tilde{\mu}_j$ s by $\frac{1}{2} \log[\tilde{\mu}_j]$:

$$\{\theta_s, \theta_t\} = 0, \quad \{\theta_s, \frac{1}{2} \log[\tilde{\mu}_t]\} = \delta_{st},$$

and

$$\{\frac{1}{2} \log[\tilde{\mu}_s], \frac{1}{2} \log[\tilde{\mu}_t]\} = \frac{1}{2} \cot\left(\frac{\theta_s - \theta_t}{2}\right).$$

The nondegeneracy of the bracket follows immediately from $\{\theta_s, \tilde{\mu}_t\} = 2\tilde{\mu}_t \delta_{st}$. Indeed, let f be a smooth function which is not constant. In that case, f depends nontrivially on at least one of the variables, say θ_1 , and hence $\frac{\partial f}{\partial \theta_1} \neq 0$ on some open set. Therefore

$$\{f, \tilde{\mu}_1\} = \frac{\partial f}{\partial \theta_1} \{\theta_1, \tilde{\mu}_1\} = 2\tilde{\mu}_1 \frac{\partial f}{\partial \theta_1} \neq 0$$

on that open set. In other words, constants are the only functions which commute with every other function, and hence the Poisson bracket is nondegenerate. \square

On this space, we are able to find the canonical coordinates for the Poisson structure defined above:

Theorem 5. *Define*

$$(4.4) \quad q_s = \tilde{\mu}_s \prod_{j \neq s} |z_s - z_j|, \quad \text{for all } 1 \leq s \leq n.$$

Then

$$(4.5) \quad \{\theta_s, q_t\} = 2q_t \delta_{st} \quad \text{and} \quad \{\theta_s, \theta_t\} = \{q_s, q_t\} = 0.$$

In other words,

$$(4.6) \quad \theta_1, \dots, \theta_n, \frac{1}{2} \log[q_1], \dots, \frac{1}{2} \log[q_n]$$

are canonical coordinates on the space of un-normalized measures $\tilde{\mu}$.

Proof. Having guessed the correct quantities which give the canonical coordinates, the proof is merely a calculation. As all the z_j 's Poisson commute, the first bracket in (4.5) follows immediately from (4.2).

So we need to compute the bracket of the q_s 's. For $s \neq t$ we have

$$\begin{aligned} \{q_s, q_t\} &= \{\tilde{\mu}_s, \tilde{\mu}_t\} \cdot \prod_{j \neq s} |z_j - z_s| \prod_{k \neq t} |z_k - z_t| \\ &\quad + \tilde{\mu}_t \prod_{j \neq s} |z_j - z_s| \cdot \{\tilde{\mu}_s, \prod_{k \neq t} |z_k - z_t|\} \\ &\quad - \tilde{\mu}_s \prod_{k \neq t} |z_k - z_t| \cdot \{\tilde{\mu}_t, \prod_{j \neq s} |z_j - z_s|\} \\ &= (\{\tilde{\mu}_s, \tilde{\mu}_t\} + T(s, t) - T(t, s)) \cdot \prod_{j \neq s} |z_j - z_s| \prod_{k \neq t} |z_k - z_t|, \end{aligned}$$

where, using (4.2) and the fact that $s \neq t$, we find that:

$$\begin{aligned} T(s, t) &= \tilde{\mu}_t \sum_{k \neq t} \frac{\{\tilde{\mu}_s, |z_k - z_t|\}}{|z_k - z_t|} \\ &= \frac{\tilde{\mu}_t}{2|z_s - z_t|^2} \cdot (\{\tilde{\mu}_s, z_s\}(\bar{z}_s - \bar{z}_t) + \{\tilde{\mu}_s, \bar{z}_s\}(z_s - z_t)) \\ &= i\tilde{\mu}_s \tilde{\mu}_t \frac{z_s \bar{z}_t - \bar{z}_s z_t}{|z_s - z_t|^2}. \end{aligned}$$

By (4.3), the antisymmetry of T and using the fact the $|z_j| = 1$ for every $1 \leq j \leq n$, we find that

$$\{\tilde{\mu}_s, \tilde{\mu}_t\} + T(s, t) - T(t, s) = 2i\tilde{\mu}_s\tilde{\mu}_t \left[\frac{z_s + z_t}{z_s - z_t} + \frac{z_s\bar{z}_t - \bar{z}_s z_t}{|z_s - z_t|^2} \right] = 0,$$

and hence

$$\{q_s, q_t\} = 0$$

for any $1 \leq s, t \leq n$. This proves the statement of the theorem. \square

Once we have the result above, the canonical coordinates for the space of normalized measures, or, equivalently, of CMV matrices, follow from a simple observation:

Corollary 4.2. *With the notations from Theorem 5 we get that*

$$(4.7) \quad \theta_1, \dots, \theta_{n-1}, \frac{1}{2} \log[r_{1,n}], \dots, \frac{1}{2} \log[r_{n-1,n}]$$

are canonical coordinates on the space of CMV matrices with fixed determinant, where

$$(4.8) \quad r_{j,k} = \frac{\mu_j}{\mu_k} \prod_{l \neq j,k} \left| \frac{z_l - z_j}{z_l - z_k} \right|.$$

Proof. The observation that justifies our claim completely is

$$r_{j,k} = \frac{q_j}{q_k}.$$

Indeed, while each q_j depends on the normalization through $\tilde{\mu}_j = c\mu_j$, their ratios do not:

$$\frac{\tilde{\mu}_j}{\tilde{\mu}_k} = \frac{c\mu_j}{c\mu_k} = \frac{\mu_j}{\mu_k}.$$

Having observed this, the claim that (4.7) are canonical coordinates follows by a moment's reflection from (4.5). \square

5. COMPATIBLE POISSON BRACKETS

In this section, we define a family of compatible (in the sense of Magri) Poisson brackets on the space of finite measures on the unit circle. Furthermore, the restrictions of all of these brackets to the manifold of probability measures represents a multi-Hamiltonian structure for the Ablowitz-Ladik equation (1.2) described in the Introduction.

Let h be a smooth function on \mathbb{C} which takes real values on S^1 . Define $\{\cdot, \cdot\}_h$ by specifying the bracket of the coordinates θ_s and q_t , $1 \leq s, t \leq n$:

$$(5.1) \quad \{\theta_s, \theta_t\}_h = \{q_s, q_t\}_h = 0 \quad \text{and} \quad \{\theta_s, q_t\}_h = 2h(e^{i\theta_s})q_t\delta_{st},$$

and then extend it in the canonical fashion:

$$(5.2) \quad \{f_1, f_2\}_h = \sum_{s=1}^n 2h(e^{i\theta_s})q_s \left[\frac{\partial f_1}{\partial \theta_s} \frac{\partial f_2}{\partial q_s} - \frac{\partial f_1}{\partial q_s} \frac{\partial f_2}{\partial \theta_s} \right].$$

Proposition 5.1. *Let $h : S^1 \rightarrow \mathbb{R}$ be a smooth, nonzero function. Then $\{\cdot, \cdot\}_h$ defined as in (5.2) (or, equivalently, (5.1)) is a Poisson bracket, and, if h is not identically zero on any arc in S^1 , then $\{\cdot, \cdot\}_h$ is nondegenerate. Furthermore, any two such brackets are compatible, in the sense that their sum is again a Poisson bracket.*

Remark 5.2. Note that for h identically equal to 1 we recover the extension of the Gelfand-Dikij bracket defined in Section 4. So the proposition claims that (5.2) defines a family of Poisson brackets compatible with the GD-bracket.

Proof. A simple calculation shows that $\{\cdot, \cdot\}_h$ obeys the Jacobi identity - note that it is sufficient to check it on the θ_s 's and q_t 's.

If h is not identically zero on any arc in S^1 , it follows immediately from the definition (5.1) that the h -bracket is nondegenerate: indeed, any smooth, nonconstant function f must depend

nontrivially on at least one of the variables, say θ_1 . Then the h -bracket of f with the conjugate variable (in this case q_1) will be nonzero:

$$\{f, q_1\}_h = \frac{\partial f}{\partial \theta_1} \{\theta_1, q_1\}_h = 2h(e^{i\theta_1}) q_1 \frac{\partial f}{\partial \theta_1} \neq 0.$$

So the only Casimirs are constant functions.

Finally, note that the newly-defined brackets are linear in h . In other words, for any h_1 and h_2 as above, the sum

$$\{\cdot, \cdot\}_{h_1} + \{\cdot, \cdot\}_{h_2} \equiv \{\cdot, \cdot\}_{h_1+h_2}$$

is, by the previous argument, also a Poisson bracket. This is exactly the definition of compatibility. \square

Going back to the $\tilde{\mu}_j$ variables, direct calculations show that:

Lemma 5.3. *In the notations used above, the h -bracket can be written in the θ and $\tilde{\mu}$ coordinates as:*

$$\{z_s, \tilde{\mu}_t\}_h = 2iz_s h(z_s) \tilde{\mu}_t \delta_{st}$$

and, for $s \neq t$,

$$\{\tilde{\mu}_s, \tilde{\mu}_t\}_h = i\tilde{\mu}_s \tilde{\mu}_t (h(z_s) + h(z_t)) \frac{z_s + z_t}{z_s - z_t}.$$

The analog of the Carathéodory function for unnormalized measures is defined, unsurprisingly, by any of the following expressions

$$(5.3) \quad \tilde{F}(z) = \int_{S^1} \frac{\zeta + z}{\zeta - z} d\tilde{\mu}(\zeta) = \sum_{j=1}^n \tilde{\mu}_j \frac{z_j + z}{z_j - z} = cF(z),$$

where, as before, $c = |\tilde{\mu}|$ is the total weight of the finite measure $\tilde{\mu}$, and F is the usual Carathéodory function associated to the probability measure $\mu = \tilde{\mu}/|\tilde{\mu}|$. Given the approach we take in this paper, it is natural to try to compute the h -bracket of \tilde{F} at two distinct points z and w in the complex plane. The calculation that will give us these formulae is straightforward enough, with the only caveat that the resulting formula will involve not only \tilde{F} , but also the function

$$\tilde{F}_h(z) = \sum_{j=1}^n h(z_j) \tilde{\mu}_j \frac{z_j + z}{z_j - z} = \int h(\zeta) \frac{\zeta + z}{\zeta - z} d\tilde{\mu}(\zeta).$$

As before, $\tilde{F}_{h=1}(z) = \tilde{F}(z)$. Then one has

$$\begin{aligned} \{\tilde{F}(z), \tilde{F}(w)\}_h &= i \frac{w+z}{w-z} (\tilde{F}(z) - \tilde{F}(w)) (\tilde{F}_h(z) - \tilde{F}_h(w)) \\ &\quad - i \tilde{F}(0) (\tilde{F}_h(z) - \tilde{F}_h(w)) - i \tilde{F}_h(0) (\tilde{F}(z) - \tilde{F}(w)) \\ &= i(w-z) \sum_{j,k=1}^n \frac{(z_j + z_k)(h(z_j) + h(z_k))(z_j z_k + zw)}{(z_j - z)(z_k - z)(z_j - w)(z_k - w)} \tilde{\mu}_j \tilde{\mu}_k \end{aligned}$$

While this formula is fairly involved and not very pretty, it simplifies greatly when restricted to the manifold of probability measures. In order to find this restriction, we need to take the reverse road to that in the previous section, and hence compute the following bracket:

$$\begin{aligned} \{c, \tilde{F}(z)\}_h &= iz \sum_{j,k=1}^n \frac{(z_j + z_k)(h(z_j) + h(z_k))}{(z_j - z)(z_k - z)} \tilde{\mu}_j \tilde{\mu}_k \\ &= i[\tilde{F}_h(z) \tilde{F}(z) - \tilde{F}_h(0) \tilde{F}(0)] \end{aligned}$$

$$\{c, F(z)\}_h = ic[F_h(z)F(z) - F_h(0)]$$

(where $\tilde{F}_h(z) = cF_h(z)$, and recall $F(0) = 1$). Finally, we obtain the following:

Theorem 6. *The restrictions of the h -brackets to the manifold of probability measures supported at n points on the unit circle are given by*

$$(5.4) \quad \{F(z), F(w)\}_h = i(F_h(z) - F_h(w)) \left[\frac{w+z}{w-z} (F(z) - F(w)) + F(z)F(w) - 1 \right].$$

which defines, for h smooth and real valued, a family of compatible Poisson brackets, that forms a multi-Hamiltonian structure for the defocusing Ablowitz-Ladik bracket.

In particular, this implies

$$(5.5) \quad \{z_s, z_t\}_h = 0, \quad \{z_s, \mu_t\}_h = 2iz_s h(z_s) \mu_t (\delta_{st} - \mu_s)$$

and, for $s \neq t$,

$$(5.6) \quad \begin{aligned} \{\mu_s, \mu_t\}_h = i\mu_s \mu_t \sum_{k \neq s, t} \mu_k & \left[\frac{(h(z_s) + h(z_t))(z_s + z_t)}{z_s - z_t} + \frac{(h(z_t) + h(z_k))(z_t + z_k)}{z_t - z_k} \right. \\ & \left. + \frac{(h(z_k) + h(z_s))(z_k + z_s)}{z_k - z_s} \right] \end{aligned}$$

Proof. The equation (5.4) follows directly from the previous formulae and from

$$\{\tilde{F}(z), \tilde{F}(w)\}_h = c^2 \{F(z), F(w)\}_h + cF(z)\{c, F(w)\}_h + cF(w)\{F(z), c\}_h.$$

The formulae for the brackets of the z_s 's and μ_t 's follow from (5.4) by using the residue theorem, exactly as in the proof of Theorem 4. \square

Finally, we close this section by identifying Hamiltonians for Ablowitz-Ladik flows in the h -brackets.

Proposition 5.4. *Let g be a polynomial, and consider the Hamiltonian on $M_n(\mathbb{C})$ defined by $\phi(X) = \text{Im Tr}(g(X))$. Then the evolution of the spectral measure $\mu = \sum_{j=1}^n \mu_j \delta_{z_j}$, $z_j = e^{i\theta_j}$, associated to a CMV matrix \mathcal{C} under this Hamiltonian in the h -bracket is given by*

$$(5.7) \quad \begin{cases} \dot{z}_j &= \{\phi, z_j\}_h = 0 \\ \dot{\mu}_j &= \{\phi, \mu_j\}_h = \mu_j \left[G(z_j)h(z_j) - \sum_{l=1}^n G(z_l)h(z_l)\mu_l \right], \end{cases}$$

where $G(z) = 2 \text{Re}(zg'(z))$. Equivalently,

$$(5.8) \quad \mu(t) = \frac{e^{G(z)h(z)t} \mu(t=0)}{|e^{G(z)h(z)t} \mu(t=0)|}.$$

Proof. Note that we can rewrite $\phi = \text{Im} \sum_{k=1}^n g(z_k)$. Then the first formula in (5.7) follows immediately from (5.5), while

$$\dot{\mu}_j = \{\phi, \mu_j\}_h = \sum_{k=1}^n \frac{\partial \phi}{\partial \theta_k} \{\theta_k, \mu_j\}_h$$

But

$$\frac{\partial \phi}{\partial \theta_k} = \text{Im}(ie^{i\theta_k} g'(e^{i\theta_k})) = \frac{1}{2} G(e^{i\theta_k}),$$

and hence

$$\begin{aligned} \dot{\mu}_j &= \sum_{k=1}^n G(e^{i\theta_k}) h(e^{i\theta_k}) \mu_j (\delta_{jk} - \mu_k) \\ &= \mu_j \left[G(e^{i\theta_j}) h(e^{i\theta_j}) - \sum_{l=1}^n G(e^{i\theta_l}) h(e^{i\theta_l}) \mu_l \right], \end{aligned}$$

as in (5.7). Finally, (5.8) follows from (5.7) by integration. \square

Recall that the Gelfand-Dikij bracket, which is the Poisson bracket associated to the Ablowitz-Ladik equation, corresponds to $h \equiv 1$, while the Hamiltonians which generate the flows in the AL hierarchy are exactly of the form $\phi(\mathcal{C}) = \text{Im Tr}(g(\mathcal{C}))$ for a polynomial g . So we find that the following holds:

Corollary 5.5. *If h is a trigonometric polynomial, then the flow generated in the h -bracket by the Hamiltonian $\phi(\mathcal{C}) = \text{Im Tr}(g(\mathcal{C}))$, where g is a polynomial, is one of Ablowitz-Ladik flows.*

Proof. Since both h and G are trigonometric polynomials, their product will have the form

$$h(z)G(z) = \sum_{j=-d}^d c_j z^j, \quad \text{where } c_{-j} = \bar{c}_j.$$

Then for $z \in S^1$

$$h(z)G(z) = 2 \text{Re} \left(\sum_{j=1}^d c_j z^j \right) + c_0 = 2 \text{Re}(z \tilde{g}'(z)) + c_0 = \tilde{G}(z) + c_0,$$

where we can choose $\tilde{g}(z) = \sum_{j=1}^d \frac{c_j}{j} z^j$ (unique up to an additive constant). If we set $\tilde{\phi}(\mathbb{C}) = \text{Im Tr}(\tilde{g}(\mathbb{C}))$, then we see that

$$\{\phi, \mu_j\}_h - \{\tilde{\phi}, \mu_j\}_1 = \mu_j \left[c_0 - \sum_{k=1}^n c_0 \mu_k \right] = 0,$$

since μ is a probability measure. This is exactly what we claimed. \square

We close this section by noting an immediate consequence of Corollary 5.5. Consider two trigonometric polynomials, h_j , $j = 1, 2$, and two polynomials g_j from which we construct Hamiltonians as before: $\phi_j(\mathcal{C}) = \text{Im Tr}(g_j(\mathcal{C}))$. Then ϕ_1 generates the same flow in $\{\cdot, \cdot\}_{h_1}$ as ϕ_2 does in $\{\cdot, \cdot\}_{h_2}$ iff

$$(5.9) \quad h_1 G_1 - h_2 G_2 = \text{const.} \in \mathbb{R},$$

where, as before, $G_j(z) = 2 \text{Re}(z g_j'(z))$ for $j = 1, 2$.

6. THE CONNECTION TO SCHUR FLOWS AND THE TODA LATTICE

Let us now consider the case where the measure $d\mu$ is symmetric with respect to complex conjugation, or what is equivalent, where all the Verblunsky parameters are real. In this case, there are an even number of eigenvalues, z_1, \dots, z_n , $n = 2N$, with the extra symmetry

$$(6.1) \quad z_{j+N} = \bar{z}_j, \quad \mu_{j+N} = \mu_j \quad \text{for } 1 \leq j \leq N.$$

For simplicity of the notation, we further assume that z_1, \dots, z_N are the eigenvalues on the upper half of the unit circle. It is a famous observation of Szegő (see [Sze, §11.5]) that the polynomials orthogonal with respect to this measure are intimately related to the polynomials orthogonal with respect to the measure $d\nu$ on $[-2, 2]$ defined by

$$(6.2) \quad \int_{S^1} f(z + z^{-1}) d\mu(z) = \int_{-2}^2 f(x) d\nu(x).$$

The recurrence coefficients for these systems of orthogonal polynomials are related by the Geronimus relations:

$$(6.3) \quad \begin{cases} b_{k+1} &= (1 - \alpha_{2k-1})\alpha_{2k} - (1 + \alpha_{2k-1})\alpha_{2k-2} \\ a_{k+1} &= \{(1 - \alpha_{2k-1})(1 - \alpha_{2k}^2)(1 + \alpha_{2k+1})\}^{1/2}. \end{cases}$$

It is an easy observation (see, for example, [Nen2]) that the second flow in the Ablowitz-Ladik hierarchy, which is generated by $\text{Im Tr } \mathcal{C}$, will preserve the property of all the Verblunsky coefficients being in $(-1, 1)$. Hence it makes sense to ask what is the flow it induces via the Geronimus relations to the a 's and b 's. A direct calculation shows that the answer is exactly the Toda flow! It is immediately clear from (1.5) that the submanifold of real Verblunsky coefficients, which we will denote by M , is in fact stable under any of the flows generated in the usual AL (or GD) bracket by the Hamiltonians $\text{Im } K_k = \frac{1}{k} \text{Im Tr}(\mathcal{C}^k)$, $k \geq 1$. We will call these *the Schur flows* (cf. [FayGek1, Gol]). But neither the Hamiltonian in question, nor the usual Ablowitz-Ladik Poisson bracket have meaningful restrictions to this submanifold, nor is it possible to find the image of the

flows generated by $\text{Im } K_k$ under the Geronimus relations by straightforward calculations. In this section we investigate the newly defined h -brackets from these points of view.

Proposition 6.1. *The h -bracket $\{\cdot, \cdot\}_h$ has a restriction to the submanifold M of probability measures on the unit circle which are symmetric with respect to complex conjugation iff $h(\bar{z}) = -h(z)$ for $z \in S^1$.*

Proof. To prove this statement, consider a set of functions on the manifold of measures supported at $n = 2N$ points defined by

$$(6.4) \quad \lambda_s = z_s + \bar{z}_s \quad \text{and} \quad \nu_s = 2\mu_s.$$

In fact, these functions with $1 \leq s \leq N$ form a set of coordinates on the submanifold of probability measures on the unit circle which are symmetric with respect to complex conjugation, and

$$(6.5) \quad \lambda_s = z_s + z_{s+N}, \quad \nu_s = \mu_s + \mu_{s+N}$$

for $1 \leq s \leq N$. Furthermore, the measure $d\nu$ on $[-2, 2]$ defined above is exactly $d\nu = \sum_{s=1}^N \nu_s \delta_{\lambda_s}$. Direct calculation using (5.5) shows that

$$\{\lambda_s, \lambda_t\}_h = 0, \quad \text{for every } 1 \leq s, t \leq N.$$

The other two types of brackets are more complicated. Decompose a general function h as $h = h_+ + h_-$, where $h_+(z) = h_+(\bar{z})$ and $h_-(z) = -h_-(\bar{z})$ for $z \in S^1$. This is equivalent to setting $2h_+(z) = h(z) + h(\bar{z})$ and $2h_-(z) = h(z) - h(\bar{z})$. Then consider $1 \leq s, t \leq N$, and use (6.5), (6.1), and (5.5) with $s, s+N, t$, and $t+N$, respectively:

$$\begin{aligned} \{\lambda_s, \nu_t\}_h &= 2iz_s h(z_s) \mu_t (\delta_{st} - \mu_s) + 2iz_s h(z_s) \mu_{t+N} (-\mu_s) \\ &\quad + 2iz_{s+N} h(z_{s+N}) \mu_t (-\mu_{s+N}) + 2iz_{s+N} h(z_{s+N}) \mu_{t+N} (\delta_{st} - \mu_{s+N}) \\ &= h_+(z_s) [2i(z_s + \bar{z}_s) \mu_t (\delta_{st} - \mu_s) - 2i(z_s + \bar{z}_s) \mu_s \mu_t] \\ &\quad + h_-(z_s) [2i(z_s - \bar{z}_s) \mu_t (\delta_{st} - \mu_s) - 2i(z_s - \bar{z}_s) \mu_s \mu_t] \\ &= 2i(z_s + \bar{z}_s) h_+(z_s) \mu_t (\delta_{st} - 2\mu_s) \\ &\quad + 2i(z_s - \bar{z}_s) h_-(z_s) \mu_t (\delta_{st} - 2\mu_s) \end{aligned}$$

The last expression is real-valued iff $h_+ \equiv 0$, or, equivalently, $h(\bar{z}) = -h(z)$. In this case, we get

$$\{\lambda_s, \nu_t\}_h = 2i(z_s - \bar{z}_s) h_-(z_s) \mu_t (\delta_{st} - 2\mu_s),$$

where the right-hand side is real for $z_s \in S^1$, and invariant under the mapping taking a probability measure to its complex conjugate.

Finally, we need to deal with the h -bracket of the ν 's. Proceeding as above, we get:

$$\{\nu_s, \nu_t\}_h = \{\mu_s, \mu_t\}_h + \{\mu_{s+N}, \mu_{t+N}\}_h + \{\mu_{s+N}, \mu_t\}_h + \{\mu_s, \mu_{t+N}\}_h$$

Group the first two terms on the right-hand side to get

$$\begin{aligned} &i\mu_s \mu_t \sum_{k \neq s, t} \mu_k \left[\frac{(h(z_s) + h(z_t))(z_s + z_t)}{z_s - z_t} + \frac{(h(z_t) + h(z_k))(z_t + z_k)}{z_t - z_k} + \frac{(h(z_k) + h(z_s))(z_k + z_s)}{z_k - z_s} \right] \\ &+ i\mu_{s+N} \mu_{t+N} \sum_{l \neq s+N, t+N} \mu_l \left[\frac{(h(z_{s+N}) + h(z_{t+N}))(z_{s+N} + z_{t+N})}{z_{s+N} - z_{t+N}} + \frac{(h(z_{t+N}) + h(z_l))(z_{t+N} + z_l)}{z_{t+N} - z_l} \right. \\ &\quad \left. + \frac{(h(z_l) + h(z_{s+N}))(z_l + z_{s+N})}{z_l - z_{s+N}} \right] \\ &= i\mu_s \mu_t \sum_{k \neq s, t} \mu_k \left[\frac{(h(z_s) + h(z_t))(z_s + z_t)}{z_s - z_t} + \frac{(h(z_{s+N}) + h(z_{t+N}))(z_{s+N} + z_{t+N})}{z_{s+N} - z_{t+N}} \right. \\ &\quad \left. + \text{cyclic permutations} \right] \end{aligned}$$

To obtain this last identity, set $l = k + N$ in the second sum on the left-hand side, and use formula (5.6) as well as the symmetry conditions (6.1). Note that we think of the indices as periodic, with period $2N$. The main observation at this point is that, for any two indices $1 \leq l, r \leq N$, we have

$$\begin{aligned} & \frac{(h(z_l) + h(z_r))(z_l + z_r)}{z_l - z_r} + \frac{(h(z_{l+N}) + h(z_{r+N}))(z_{l+N} + z_{r+N})}{z_{l+N} - z_{r+N}} \\ &= (h_+(z_l) + h_+(z_r)) \left[\frac{z_l + z_r}{z_l - z_r} + \frac{\bar{z}_l + \bar{z}_r}{\bar{z}_l - \bar{z}_r} \right] + (h_-(z_l) + h_-(z_r)) \left[\frac{z_l + z_r}{z_l - z_r} - \frac{\bar{z}_l + \bar{z}_r}{\bar{z}_l - \bar{z}_r} \right] \end{aligned}$$

But it is a simple observation that, for $z, w \in S^1$,

$$\frac{\bar{z} + \bar{w}}{\bar{z} - \bar{w}} = -\frac{z + w}{z - w},$$

and hence the expression above equals

$$2(h_-(z_l) + h_-(z_r)) \frac{z_l + z_r}{z_l - z_r}.$$

In particular, this means that we can work our way backwards to the original expressions for the four h -brackets, only with h replaced by h_- . Note that, for any $z \neq w \in S^1$, $i \frac{z+w}{z-w} \in \mathbb{R}$ and

$$\frac{(h_-(z) + h_-(w))(z + w)}{z - w} = \frac{(h_-(\bar{z}) + h_-(\bar{w}))(\bar{z} + \bar{w})}{\bar{z} - \bar{w}}.$$

But this immediately implies that, even though the bracket $\{\nu_s, \nu_t\}_h$ is still the sum of four complicated formulae, it is real-valued and invariant under the mapping taking a probability measure to its complex conjugate, which completes the proof. \square

Now consider a function h obeying

$$h(e^{-i\theta}) = -h(e^{i\theta}),$$

and restrict the bracket $\{\cdot, \cdot\}_h$ to the subspace of real Verblunsky coefficients. We want to write this restriction as a combination of the (compatible) Poisson brackets for the Toda lattice found in [FayGek2]. To avoid confusion, we will go back to the notation from Section 3. Hence we will start denoting the h -brackets by $\{\cdot, \cdot\}_h^{(1)}$, the superscript denoting the fact that this is a Poisson bracket compatible with the Gelfand-Dikij bracket. By contrast, we will later on be interested in some of the Poisson brackets compatible with the Lie-Poisson bracket, which were originally introduced in [FayGek2], and which we will denote here by $\{\cdot, \cdot\}_H^{(0)}$, for some function H .

In order to achieve this, we must first relate the Carathéodory function of a measure $d\mu$ on the circle which is invariant under complex conjugation to the m -function of the associated measure $d\nu$ on $[-2, 2]$. A simple calculation (see, for example, [Sim1]) show that, for $z \in \mathbb{C} \setminus \{0\}$, we have

$$(6.6) \quad F(z) = -F\left(\frac{1}{z}\right) = \left(z - \frac{1}{z}\right) \cdot m\left(z + \frac{1}{z}\right),$$

and

$$(6.7) \quad F_h(z) = F_h\left(\frac{1}{z}\right) = im_H\left(z + \frac{1}{z}\right),$$

where

$$(6.8) \quad H(2 \cos \theta) = 2 \sin(\theta) h(e^{i\theta}),$$

h obeys $h(e^{-i\theta}) = -h(e^{i\theta})$ as above, and m_H is defined by

$$(6.9) \quad m_H(\lambda) = \int_{-2}^2 H(t) \frac{1}{t - \lambda} d\nu(t).$$

Note that, for $z \in S^1$, H is defined by

$$H(z + \bar{z}) = (z - \bar{z})h(z),$$

which is exactly the type of combination that has already appeared in the proof of Proposition 6.1.

We wish to find the h -bracket of the m function and compare it to the ones in [FayGek2]. To do so, we insert (6.6) into (5.4), but we must not lose sight of the symmetry inherent to the situation. In this case, we use the fact that

$$(6.10) \quad 4\left(z - \frac{1}{z}\right)\left(w - \frac{1}{w}\right)\{m(z + \frac{1}{z}), m(w + \frac{1}{w})\}_h^{(1)} = \{F(z) - F(\frac{1}{z}), F(w) - F(\frac{1}{w})\}_h^{(1)}$$

Using (6.6), we get that

$$\begin{aligned} \{F(z) - F(\frac{1}{z}), F(w) - F(\frac{1}{w})\}_h^{(1)} &= \{F(z), F(w)\}_h^{(1)} - \{F(\frac{1}{z}), F(w)\}_h^{(1)} \\ &\quad - \{F(z), F(\frac{1}{w})\}_h^{(1)} + \{F(\frac{1}{z}), F(\frac{1}{w})\}_h^{(1)} \\ &= (F_h(z) - F_h(w)) \cdot (B(z, w) - 4F(z)F(w)) \end{aligned}$$

where

$$\begin{aligned} B(z, w) &= -\frac{w+z}{w-z}(F(z) - F(w)) + \frac{w+\frac{1}{z}}{w-\frac{1}{z}}(-F(z) - F(w)) \\ &\quad + \frac{\frac{1}{w}+z}{\frac{1}{w}-z}(F(z) + F(w)) - \frac{\frac{1}{w}+\frac{1}{z}}{\frac{1}{w}-\frac{1}{z}}(-F(z) + F(w)) \\ &= 4z \frac{1-w^2}{(z-w)(1-zw)} F(z) - 4w \frac{1-z^2}{(z-w)(1-zw)} F(w) \\ &= 4\left(z - \frac{1}{z}\right)\left(w - \frac{1}{w}\right) \frac{m(z + \frac{1}{z}) - m(w + \frac{1}{w})}{(z + \frac{1}{z}) - (w + \frac{1}{w})} \end{aligned}$$

Putting it all together and replacing the combination $z + \frac{1}{z}$ by a general $\lambda \in \mathbb{C} \setminus \text{spec}(d\nu)$, we get that

$$\{m(\lambda), m(\xi)\}_h^{(1)} = (m_H(\lambda) - m_H(\xi)) \cdot \left[\frac{m(\lambda) - m(\xi)}{\lambda - \xi} - m(\lambda)m(\xi) \right]$$

Let us recall that, for $k \geq 0$, Faybusovich and Gekhtman defined a Poisson bracket on the space of measures supported on $[-2, 2]$. This k -bracket can be written down in terms of various coordinates, but here we concentrate on its expression in terms of the associated m -functions:

$$(6.11) \quad \{m(\lambda), m(\xi)\}_k = ((\lambda^k m(\lambda))_- - (\xi^k m(\xi))_-) \cdot \left[\frac{m(\lambda) - m(\xi)}{\lambda - \xi} - m(\lambda)m(\xi) \right]^5,$$

where, for a meromorphic function $r : \mathbb{C} \rightarrow \mathbb{C}$ with $r(\lambda) = \sum_{l=-\infty}^N c_l \lambda^l$, $N < \infty$, we set

$$(r(\lambda))_+ = \sum_{l=0}^N c_l \lambda^l \quad \text{and} \quad (r(\lambda))_- = r(\lambda) - (r(\lambda))_+ = \sum_{l=-\infty}^{-1} c_l \lambda^l.$$

Note that for any $k \geq 0$ we have

$$\left(\frac{t^k - \lambda^k}{t - \lambda} \right)_- = \left(\sum_{j=0}^{k-1} t^j \lambda^{k-j-1} \right)_- = 0,$$

and hence

$$\left(\frac{\lambda^k}{t - \lambda} \right)_- = \left(\frac{t^k}{t - \lambda} \right)_- = \frac{t^k}{t - \lambda}.$$

So by integrating, we get that

$$(H(\lambda)m(\lambda))_- = m_H(\lambda),$$

where H is a polynomial and m_H is defined by (6.9). In other words, we have proved the following

Proposition 6.2. *If h is a smooth, real-valued function on the unit circle S^1 such that $h(\bar{z}) = -h(z)$, then the restriction of the bracket $\{\cdot, \cdot\}_h^{(1)}$ to the submanifold of probability measures invariant under complex conjugation is given, for $\lambda \neq \xi \in \mathbb{C} \setminus \mathbb{R}$, by*

$$(6.12) \quad \{m(\lambda), m(\xi)\}_h^{(1)} = (m_H(\lambda) - m_H(\xi)) \cdot \left[\frac{m(\lambda) - m(\xi)}{\lambda - \xi} - m(\lambda)m(\xi) \right],$$

⁵We use here the notation of [FayGek2] for the compatible brackets.

where H , m and m_H are defined as above.

The right-hand side defines a Poisson bracket $\{\cdot, \cdot\}_H^{(0)}$ on the manifold of probability measures supported at N points on $[-2, 2]$ which is compatible with the Toda lattice (i.e. restriction of the Lie-Poisson) bracket. Furthermore, if h is a trigonometric polynomial, then H is a polynomial and $\{\cdot, \cdot\}_H^{(0)}$ is a linear combination of the compatible brackets (6.11) of [FayGek2].

While we could try to write down the general formula for these brackets, we will limit ourselves to investigating the simplest case, which is

$$h(e^{i\theta}) = 2 \sin(\theta).$$

Then we get that $H(2 \cos(\theta)) = 4 \sin(\theta)^2$, or, equivalently,

$$H(t) = 4 - t^2, \quad \text{for } t \in [-2, 2].$$

Therefore we find that the restriction of the Poisson bracket $\{\cdot, \cdot\}_{2 \sin}^{(1)}$ to the space of real Verblunsky coefficients coincides, under the Geronimus relations, to the Poisson bracket $\{\cdot, \cdot\}_H^{(0)} = 4\{\cdot, \cdot\}_0 - \{\cdot, \cdot\}_2$.

We close this paper by identifying the Hamiltonians defining certain Ablowitz-Ladik flows with Toda hierarchy Hamiltonians. Recall that the usual AL bracket is $\{\cdot, \cdot\}_{h_1}$ for $h_1 \equiv 1$, and let $h_2(\bar{z}) = -h_2(z)$ for $z \in S^1$, as in Proposition 6.1. Consider a polynomial with real coefficients, g_1 , and note that the Hamiltonian $\phi_1(\mathcal{C}) = \text{Im Tr}_1(\mathcal{C})$ is a linear combination, with real coefficients, of the flows $\text{Im } K_k$ and hence it generates a flow under which M is stable. If g_2 is another polynomial and $\phi_2(\mathcal{C}) = \text{Im Tr}_2(\mathcal{C})$, then from (5.9) we get that if

$$(6.13) \quad G_1(z) - h_2(z)G_2(z) \equiv \text{const.} \in \mathbb{R},$$

then the two flows coincide:

$$\{\phi_1, \cdot\}_{GD} \equiv \{\phi_2, \cdot\}_{h_2}.$$

Here, as in Section 5, $G_j(z) = 2 \text{Re}(zg'_j(z))$. Let us make a few remarks on (6.13):

- Since $g_1(z) = \sum_{j=0}^d c_j z^j$ with $c_j \in \mathbb{R}$, we get $G_1(z) = \sum_{j=1}^d j c_j (z^j + \bar{z}^j)$ and so for $z \in S^1$, $G_1(\bar{z}) = G_1(z)$. Since we imposed $h_2(\bar{z}) = -h_2(z)$, we must have that $G_2(\bar{z}) = -G_2(z)$, or, if we work as in the proof of Corollary 5.5, the coefficients of g_2 are purely imaginary, up to an additive constant which we will ignore since it does not influence the flow. So the first observation is that

$$(6.14) \quad \phi_2(\mathcal{C}) = \text{Re Tr}(-ig_2(\mathcal{C}))$$

is a linear combination with real coefficients of the Hamiltonians $\text{Re } K_k$, and hence has a nontrivial restriction to the submanifold M .

- A straightforward count of the parameters in (6.13) shows that it is not true that given g_1 , we can always find h_2 and G_2 with the required properties and which satisfy (6.13). Indeed, without loss of generality we may assume that g_1 , h_2 and G_2 are monic. Then the right-hand side of (6.13) is determined by $\deg(h_2) - 1 + \deg(G_2) - 1 = \deg(G_1) - 2$ parameters, while the left-hand side imposes $\deg(G_1) - 1$ conditions. In other words, we need an extra degree of freedom. Without going into too many details, let us mention that one way to deal with this problem is to allow the Schur flow in question to be modified by a constant multiple of the first Schur flow: for any g_1 monic of degree at least 2 and with real coefficients, there exist a real constant c , and monic h_2 and g_2 as above so that $\tilde{G}_1(z) = 2 \text{Re}(zg'_1(z) - cz)$, h_2 and G_2 obey (6.13). In particular, the restriction to the submanifold M of a Schur flow whose G_1 obeys (6.13) for some appropriate h_2 and G_2 is a Hamiltonian flow in the Poisson bracket $\{\cdot, \cdot\}|_M$.⁶

- Finally, we must understand the image through the Geronimus relations of the restriction to M of Hamiltonians given by $\text{Re}(g(\mathcal{C}))$, with g polynomial with real coefficients. But this can be obtained immediately from Proposition B.3 of Killip and Nenciu [KilNen1], which shows that, if $\alpha_j \in (-1, 1)$ for all j , then $\mathcal{C} + \mathcal{C}^T$ is unitarily equivalent to a direct sum of two Jacobi matrices, J and \tilde{J} , where entries of J are related to the Verblunsky coefficients defining \mathcal{C} by the Geronimus relations. Furthermore, the spectral measure for J w.r.t. the vector $e_1 = [1 0 \dots]^T \in \mathbb{R}^n$ is $d\nu$,

⁶Since $h_2(\bar{z}) = -h_2(z)$, we know from Proposition 6.1 that M is a Poisson submanifold in this Poisson structure.

while the spectral measure of \tilde{J} w.r.t. e_1 is $d\tilde{\nu}(x) = \frac{1}{2(1-\alpha_0^2)(1-\alpha_1)}(4-x^2)d\nu(x)$. In particular, we see that the two measures ν and $\tilde{\nu}$ have the same support, or, equivalently, J and \tilde{J} have the same eigenvalues. For example,

$$\operatorname{Re} \operatorname{Tr} \mathcal{C} = \frac{1}{2} \operatorname{Tr}(\mathcal{C} + \mathcal{C}^T) = \frac{1}{2} \operatorname{Tr}(J \oplus \tilde{J}) = \operatorname{Tr}(J).$$

A slightly more careful analysis will show that for any $k \geq 1$ there exists a monic polynomial g_k , with real coefficients, so that

$$\operatorname{Re} \operatorname{Tr}(\mathcal{C}^k) \upharpoonright_M = \operatorname{Tr}(g_k(J)).$$

But the Hamiltonians on the right-hand side are exactly Toda hierarchy Hamiltonians.

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